UMTG–198

Factorization of Multiparticle Scattering in the Heisenberg Spin Chain

Anastasia Doikou, Luca Mezincescu and Rafael I. Nepomechie

Physics Department, P.O. Box 248046, University of Miami

Coral Gables, FL 33124 USA

Abstract

We give an explicit proof within the framework of the Bethe Ansatz/string hypothesis of the factorization of multiparticle scattering in the antiferromagnetic spin-$\frac{1}{2}$ Heisenberg spin chain, for the case of 3 particles.

1 Introduction

For models in $1+1$ dimensions, there are general arguments [1] that integrability (i.e., the existence of nontrivial local integrals of motion) implies factorized scattering. This fact is the basis of the "bootstrap" approach for determining S matrices for integrable quantum field theories.

On the other hand, for integrable quantum spin chains, it is possible to compute the excitations' exact S matrices directly from the Bethe Ansatz equations, without explicitly assuming factorization [2] - [5]. Therefore, it should be possible to explicitly demonstrate within this framework the factorizability of multiparticle scattering. The purpose of this note is to provide such a demonstration for the antiferromagnetic spin-$\frac{1}{2}$ Heisenberg spin chain

$$H = \frac{1}{4}\sum_{n=1}^{N} \left(\vec{\sigma}_n \cdot \vec{\sigma}_{n+1} - 1\right) , \qquad \vec{\sigma}_{N+1} \equiv \vec{\sigma}_1 , \tag{1}$$

which is the prototypical integrable quantum spin chain. For simplicity, we restrict our attention to the case of 3-body scattering, which is the first nontrivial case to exhibit factorization. We remark, following Faddeev and Takhtajan [3], that states with an odd number of excitations appear in the sector of the model with $N =$ odd.[1]

The outline of this paper is as follows. In Sec. 2 we review the Bethe Ansatz/string hypothesis description of multiparticle excited states of the Heisenberg spin chain. We compute the three-particle S matrix in Sec. 3; and we show that it is factorizable into a product of two-particle S matrices in Sec. 4. Finally, in Sec. 5 we discuss the prospects of extending this analysis to the case of more than three particles. In the Appendix we briefly review the two-particle S matrix.

2 Excited States

In this section we review the Bethe Ansatz/string hypothesis description of excited states of the Heisenberg Hamiltonian (1). Adopting the string hypothesis, the Bethe ansatz equations lead to the following equations for the real centers λ_α^n of the strings (see, e.g., Refs. [4], [7]):

$$h_n(\lambda_\alpha^n) = J_\alpha^n , \tag{2}$$

[1] For another recent investigation which exploits the odd sector of this model, see Ref. [6].

where $\alpha = 1, \cdots, M_n$ (M_n is the number of strings of length n) and $n = 1, \cdots, \infty$. The so-called counting function $h_n(\lambda)$ is defined by

$$h_n(\lambda) = \frac{1}{2\pi} \left\{ N q_n(\lambda) - \sum_{m=1}^{\infty} \sum_{\beta=1}^{M_m} \Xi_{nm}(\lambda - \lambda_\beta^m) \right\} , \tag{3}$$

$\Xi_{nm}(\lambda)$ is given by

$$\Xi_{nm}(\lambda) = (1 - \delta_{nm}) q_{|n-m|}(\lambda) + 2 q_{|n-m|+2}(\lambda) + \cdots + 2 q_{n+m-2}(\lambda) + q_{n+m}(\lambda) , \tag{4}$$

and $q_n(\lambda)$ is the odd monotonic-increasing function defined by

$$q_n(\lambda) = \pi + i \log e_n(\lambda) , \qquad -\pi < q_n(\lambda) \le \pi , \tag{5}$$

where

$$e_n(\lambda) = \frac{\lambda + \frac{in}{2}}{\lambda - \frac{in}{2}} . \tag{6}$$

Moreover $\{J_\alpha^n\}$ are integers or half-odd integers which satisfy

$$- J_{max}^n \le J_\alpha^n \le J_{max}^n , \tag{7}$$

where J_{max}^n is given by

$$J_{max}^n = \frac{1}{2} (N + M_n - 1) - \sum_{m=1}^{\infty} min(m, n) \ M_m . \tag{8}$$

We regard $\{J_\alpha^n\}$ as "quantum numbers" which parametrize the Bethe Ansatz states. For every set $\{J_\alpha^n\}$ in the range given by Eq. (7) (no two of which are identical), we assume that there is a unique solution $\{\lambda_\alpha^n\}$ (no two of which are identical) of Eq. (2).

The spin eigenvalues of the Bethe Ansatz states are given by

$$S = S^z = \frac{N}{2} - \sum_{n=1}^{\infty} n M_n . \tag{9}$$

The ground state is a "filled Fermi sea," with $M_1 = \frac{N}{2}$ and $M_n = 0$ for $n > 1$. The number of holes (excitations) ν in a Bethe Ansatz state is given by

$$\begin{aligned} \nu &= \text{number of vacancies for } J_\alpha^1\text{'s} - \text{ number of } J_\alpha^1\text{'s} \\ &= \left(2 J_{max}^1 + 1\right) - M_1 . \end{aligned} \tag{10}$$

A hole with rapidity λ has energy $\varepsilon(\lambda)$ and momentum $p(\lambda)$ given by

$$\varepsilon(\lambda) = \frac{\pi}{2\cosh \pi\lambda}\,, \qquad p(\lambda) = \tan^{-1}(\sinh \pi\lambda) - \frac{\pi}{2}\,, \tag{11}$$

respectively, and has spin $\frac{1}{2}$.

We shall focus on the following two classes of Bethe Ansatz states:

(a) $M_1 = \frac{N}{2} - \frac{\nu}{2}$ and $M_n = 0$ for $n > 1$. This state has ν holes and $S = S^z = \frac{\nu}{2}$.

(b) $M_1 = \frac{N}{2} - \frac{\nu}{2} - 1$, $M_2 = 1$ and $M_n = 0$ for $n > 2$. This state has ν holes and one 2-string, with $S = S^z = \frac{\nu}{2} - 1$. This state has a multiplicity given by

$$\text{multiplicity} = \begin{pmatrix} \text{number of vacancies for } J^2_\alpha\text{'s} \\ \text{number of } J^2_\alpha\text{'s} \end{pmatrix} = \begin{pmatrix} 2J^2_{max} + 1 \\ M_2 \end{pmatrix} = \nu - 1\,. \tag{12}$$

We define the density $\sigma(\lambda)$ of roots and holes

$$\sigma(\lambda) = \frac{1}{N}\frac{dh_1(\lambda)}{d\lambda}\,, \tag{13}$$

which shall play an important role in the following. Passing from the sum in $h_1(\lambda)$ to an integral, we obtain an integral equation whose solution is given by [2]

$$\sigma(\lambda) = s(\lambda) + \frac{1}{N}r(\lambda)\,, \tag{14}$$

where

$$r_{(a)}(\lambda) = \sum_{\alpha=1}^{\nu} J(\lambda - \tilde{\lambda}_\alpha)\,, \tag{15}$$

$$r_{(b)}(\lambda) = r_{(a)}(\lambda) - a_1(\lambda - \lambda_0)\,, \tag{16}$$

for the states (a) and (b), respectively; $\tilde{\lambda}_1, \cdots, \tilde{\lambda}_\nu$ are the hole rapidities; and $\lambda_0 \equiv \lambda^2_1$ is the position of the center of the 2-string, which we determine below. Moreover, we use the notations

$$a_n(\lambda) = \frac{1}{2\pi}\frac{dq_n(\lambda)}{d\lambda}\,, \tag{17}$$

[2]Since σ depends also on the hole rapidities, the notation $\sigma(\lambda, \tilde{\lambda}_1, \cdots, \tilde{\lambda}_\nu)$ would be more accurate. However, following the usual practice, we suppress the dependence on the hole rapidities.

$$s(\lambda) = \frac{1}{2\cosh \pi\lambda} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ e^{-i\omega\lambda}\ \frac{e^{-|\omega|/2}}{1+e^{-|\omega|}}\,, \tag{18}$$

$$J(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ e^{-i\omega\lambda}\ \frac{e^{-|\omega|}}{1+e^{-|\omega|}}\,. \tag{19}$$

In order to calculate the position λ_0 of the center of the 2-string in state (b), we observe that $h_2(\lambda_0) = J_1^2$. Passing from the sum to the integral, we eventually obtain

$$\sum_{\alpha=1}^{\nu} q_1(\lambda_0 - \tilde{\lambda}_\alpha) = 2\pi J_1^2\,. \tag{20}$$

By exponentiating and also recalling Eq. (8), we obtain the desired result

$$\prod_{\alpha=1}^{\nu} e_1(\lambda_0 - \tilde{\lambda}_\alpha) = 1\,. \tag{21}$$

The last equation can be solved for the position λ_0 of the center of the 2-string in terms of the hole rapidities. For example, for the particular case of two holes ($\nu = 2$), we obtain the well-known result

$$\lambda_0 = \frac{1}{2}\left(\tilde{\lambda}_1 + \tilde{\lambda}_2\right)\,. \tag{22}$$

For $\nu = 3$, there are two solutions:

$$\lambda_0^{\pm} = \frac{1}{3}\left(\tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3 \pm \sqrt{A + \frac{3}{4}}\right)\,, \tag{23}$$

where

$$A = \tilde{\lambda}_1(\tilde{\lambda}_1 - \tilde{\lambda}_2) + \tilde{\lambda}_2(\tilde{\lambda}_2 - \tilde{\lambda}_3) + \tilde{\lambda}_3(\tilde{\lambda}_3 - \tilde{\lambda}_1)\,. \tag{24}$$

3 Three-particle S matrix

We now restrict our attention to states with 3 excitations. Recall that the excitations have spin $\frac{1}{2}$, and that $\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} = \frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}$. The state with $S = S^z = \frac{3}{2}$ can evidently be identified as the Bethe Ansatz state (a) introduced in the previous section with $\nu = 3$.

Moreover, the two states with $S = S^z = \frac{1}{2}$ are the Bethe Ansatz states (b). Indeed, the multiplicity factor given in Eq. (12) for $\nu = 3$ is equal to 2.

Following Refs. [2], [5], we define the S matrix $S(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3)$ (acting in $C^2 \bigotimes C^2 \bigotimes C^2$) for a hole of rapidity $\tilde{\lambda}_1$ scattering with holes of rapidities $\tilde{\lambda}_2$ and $\tilde{\lambda}_3$ by the momentum quantization condition

$$\left(e^{ip(\tilde{\lambda}_1)N}\ S(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3) - 1\right) |\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3\rangle = 0\,, \tag{25}$$

where the hole momentum $p(\lambda)$ is given in Eq. (11).

We shall compute the eigenvalues of S. Let $S_{(a)}$, $S_{(b)}$ be the eigenvalues of S corresponding to states (a), (b), respectively. For state (a),

$$e^{ip(\tilde{\lambda}_1)N}\ S_{(a)} = 1\,, \tag{26}$$

and similarly for (b).

We next derive a relation between $p(\tilde{\lambda}_1)$ and the quantity $r(\lambda)$ which characterizes the distribution of roots and holes for a state. From Eqs. (11), (13), (14) and (18), we obtain

$$\frac{1}{2\pi}\frac{dp}{d\lambda} + \frac{1}{N} r = \frac{1}{N}\frac{dh_1}{d\lambda}\,. \tag{27}$$

Integrating from $-\infty$ to $\tilde{\lambda}_1$ and exponentiating, we obtain the desired relation

$$e^{ip(\tilde{\lambda}_1)N}\ e^{i2\pi\int_{-\infty}^{\tilde{\lambda}_1} r(\lambda)\ d\lambda}\ e^{i2\pi\left[h_1(-\infty) - \tilde{J}_1\right]}\ e^{-iNp(-\infty)} = 1\,, \tag{28}$$

where $h_1(\tilde{\lambda}_1) = \tilde{J}_1$.

Comparing this relation with Eq. (26), we see that

$$S_{(a)} \sim e^{i2\pi\int_{-\infty}^{\tilde{\lambda}_1} r_{(a)}(\lambda)\ d\lambda}\,. \tag{29}$$

Using the explicit expression for $r_{(a)}(\lambda)$ given in Eq. (15) with $\nu = 3$, we conclude that $S_{(a)}$ is given (up to a rapidity-independent phase factor) by

$$S_{(a)} = S_t(\tilde{\lambda}_1 - \tilde{\lambda}_2)\ S_t(\tilde{\lambda}_1 - \tilde{\lambda}_3)\,, \tag{30}$$

where

$$S_t(\lambda) = \frac{\Gamma\left(1 + \frac{i\lambda}{2}\right)\Gamma\left(\frac{1}{2} - \frac{i\lambda}{2}\right)}{\Gamma\left(1 - \frac{i\lambda}{2}\right)\Gamma\left(\frac{1}{2} + \frac{i\lambda}{2}\right)}\,. \tag{31}$$

Since the state (a) has spin $S = \frac{3}{2}$, the eigenvalue $S_{(a)}$ is 4-fold degenerate.

Although we have determined $S_{(a)}$ only up to a rapidity-independent phase factor, we can compute the ratio $S_{(b)}/S_{(a)}$ exactly:

$$\frac{S_{(b)}}{S_{(a)}} = e^{i2\pi \int_{-\infty}^{\tilde{\lambda}_1} \left[r_{(b)}(\lambda) - r_{(a)}(\lambda)\right]\, d\lambda}\ e^{i2\pi \left[h_1^{(b)}(-\infty) - h_1^{(a)}(-\infty)\right]}\ e^{-i2\pi \left(\tilde{J}_1^{(b)} - \tilde{J}_1^{(a)}\right)} = e_1(\tilde{\lambda}_1 - \lambda_0)\,, \qquad (32)$$

where we have used Eqs. (2), (8), and (16). (Each of the last two factors in Eq. (32) equals -1.) We conclude that

$$S_{(b)}^{\pm} = e_1(\tilde{\lambda}_1 - \lambda_0^{\pm})\ S_{(a)}\,, \qquad (33)$$

where $\lambda_0^{\pm}$ are given by Eq. (23). Since the states (b) have $S = \frac{1}{2}$, the eigenvalues $S_{(b)}^{\pm}$ are each 2-fold degenerate.

In summary, the S matrix $S(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3)$ has eigenvalues $S_{(a)}$ (4-fold degenerate) and $S_{(b)}^{\pm}$ (each 2-fold degenerate).

4 Factorization

We now demonstrate that the three-particle S matrix is factorizable. That is, we show that the S matrix is equal (up to a unitary transformation and a rapidity-independent phase) to a product of R matrices

$$S(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3) = R_{12}(\tilde{\lambda}_1 - \tilde{\lambda}_2)\ R_{13}(\tilde{\lambda}_1 - \tilde{\lambda}_3)\,, \qquad (34)$$

where $R(\lambda)$ is the two-particle S matrix, which can be written as (see Appendix)

$$R(\lambda) = S_t(\lambda) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(1 + e_1(\frac{\lambda}{2})) & \frac{1}{2}(1 - e_1(\frac{\lambda}{2})) & 0 \\ 0 & \frac{1}{2}(1 - e_1(\frac{\lambda}{2})) & \frac{1}{2}(1 + e_1(\frac{\lambda}{2})) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (35)$$

where $S_t(\lambda)$ is given by Eq. (31).

We prove this by showing that the eigenvalues of the RHS of Eq. (34) coincide with those of the LHS. Indeed, by explicit calculation, we find that the eigenvalues of the 8×8 matrix $R_{12}(\tilde{\lambda}_1 - \tilde{\lambda}_2) R_{13}(\tilde{\lambda}_1 - \tilde{\lambda}_3)$ are $S_{(a)}$ (4-fold degenerate) and $\alpha^{\pm} S_{(a)}$ (each 2-fold degenerate), where

$$\alpha^{\pm} = \frac{1}{8}(B \pm \sqrt{B^2 - 64\gamma_1\gamma_2})\,, \qquad (36)$$

and

$$B = 1 + 3\gamma_1 + 3\gamma_2 + \gamma_1\gamma_2\,, \tag{37}$$

with

$$\begin{aligned}\gamma_1 &= e_1(\frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{2})\,,\\ \gamma_2 &= e_1(\frac{\tilde{\lambda}_1 - \tilde{\lambda}_3}{2})\,.\end{aligned} \tag{38}$$

By virtue of the algebraic identity[3]

$$e_1(\tilde{\lambda}_1 - \lambda_0^{\pm}) = \frac{1}{8}(B \pm \sqrt{B^2 - 64\gamma_1\gamma_2})\,, \tag{39}$$

(where $\lambda_0^{\pm}$ are given by Eq. (23)), we see that

$$S_{(b)}^{\pm} = \alpha^{\pm} S_{(a)}\,. \tag{40}$$

Therefore, the eigenvalues of $R_{12}(\tilde{\lambda}_1 - \tilde{\lambda}_2) R_{13}(\tilde{\lambda}_1 - \tilde{\lambda}_3)$ indeed coincide with those of $S(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3)$, and Eq. (34) is proved.

Note that by combining Eqs. (25) and (34), we obtain

$$\left(e^{ip(\tilde{\lambda}_1)N} R_{12}(\tilde{\lambda}_1 - \tilde{\lambda}_2) R_{13}(\tilde{\lambda}_1 - \tilde{\lambda}_3) - 1\right) |\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3\rangle = 0\,, \tag{41}$$

which is Yang's formula [9], [10] for the case of 3 particles.

5 Discussion

We have seen that for the case of three holes ($\nu = 3$), the proof of factorizability of the S matrix rests on an algebraic identity given by Eq. (39). We expect that it should be possible to extend this analysis to the case $\nu > 3$. The corresponding identities will presumably be more complicated. Indeed, for sufficiently large ν, the algebraic equation (21) which determines λ_0 will be of degree higher than 4. Nevertheless, the problem of determining the eigenvalues of $R_{12}(\tilde{\lambda}_1 - \tilde{\lambda}_2) R_{13}(\tilde{\lambda}_1 - \tilde{\lambda}_3) \cdots R_{1\nu}(\tilde{\lambda}_1 - \tilde{\lambda}_\nu)$ is equivalent to diagonalizing an inhomogeneous Heisenberg spin chain transfer matrix, and hence, can be solved for arbitrary ν by Bethe Ansatz. (See, e.g., Ref. [10].)

[3] Similar identities have appeared in connection with scattering of excitations in open spin chains [8].

6 Acknowledgments

We thank H. de Vega and F. Essler for discussions. This work was supported in part by the National Science Foundation under Grant PHY-9507829.

7 Appendix: Two-particle S matrix

The two-particle S matrix $R(\lambda)$ was first computed by Faddeev and Takhtajan [4]. It can easily be obtained from Secs. 2 and 3 of the present paper. Indeed, the state $S = S^z = 1$ (triplet) and the singlet state $S = S^z = 0$ correspond to states (a) and (b), respectively, with $\nu = 2$. Let S_t and S_s be the eigenvalues of $R(\lambda)$ corresponding to the triplet and singlet states. S_t is given by Eq. (31) (see Eqs. (15) and (29)), and S_s is given by

$$S_s(\lambda) = e_1(\frac{\lambda}{2})\ S_t(\lambda) \tag{42}$$

(see Eqs. (22) and (32)).

We organize these eigenvalues into a 4×4 matrix as follows: by $SU(2)$ symmetry, $R(\lambda)$ has the general form

$$R(\lambda) = \alpha I + \beta P\,, \tag{43}$$

where I is the unit matrix, and P is the permutation matrix. The eigenvalues of this matrix are $\alpha + \beta$ (3-fold degenerate) and $\alpha - \beta$, which we identify as S_t and S_s, respectively. It follows that

$$\begin{aligned} \alpha &= \frac{1}{2}(S_t + S_s)\,, \\ \beta &= \frac{1}{2}(S_t - S_s)\,. \end{aligned} \tag{44}$$

References

[1] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. *120* (1979) 253.

[2] V.E. Korepin, Theor. Math. Phys. *76* (1980) 165; V.E. Korepin, G. Izergin and N.M. Bogoliubov, *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz* (Cambridge University Press, 1993).

[3] L.D. Faddeev and L.A. Takhtajan, Phys. Lett. *85A* (1981) 375.

[4] L.D. Faddeev and L.A. Takhtajan, J. Sov. Math. *24* (1984) 241.

[5] N. Andrei and C. Destri, Nucl. Phys. *B231* (1984) 445.

[6] A. Doikou, L. Mezincescu and R.I. Nepomechie, "Simplified Calculation of Boundary S Matrices," hep-th/9705187, J. Phys. *A*, in press.

[7] L. Mezincescu and R.I. Nepomechie, in *New Developments of Integrable Systems and Long-Ranged Interaction Models*, ed. by M-L Ge and Y-S Wu (World Scientific, 1995).

[8] M. Grisaru, L. Mezincescu and R.I. Nepomechie, J. Phys. *A28* (1995) 1027.

[9] C.N. Yang, Phys. Rev. Lett. *19* (1967) 1312.

[10] A.B. Zamolodchikov and Al.B. Zamolodchikov, Nucl. Phys. *B379* (1992) 602.